

# The minimal context for local boundedness in topological vector spaces

M.D. Voisei

## Abstract

The local boundedness of classes of operators is analyzed on different subsets directly related to their Fitzpatrick functions and characterizations of the topological vector spaces for which that local boundedness holds is given in terms of the uniform boundedness principle. For example the local boundedness of a maximal monotone operator on the algebraic interior of its domain convex hull is a characteristic of barreled locally convex spaces.

## 1 Introduction

The local boundedness of a monotone operator defined on an open set of a Banach space was first intuited by Kato in [5] while performing a comparison of (sequential) demicontinuity and hemicontinuity. Under a Banach space settings, the first result concerning the local boundedness of monotone operators appears in 1969 and is due to Rockafellar [6, Theorem 1, p. 398]. In 1972 in [3], the local boundedness of monotone-type operators is proved under a Fréchet space context. In 1988 the local boundedness of a monotone operator defined in a barreled normed space is proved in [1] on the algebraic interior of the domain. The authors of [1] call their assumptions “minimal” but they present no argument about the minimality of their hypotheses or in what sense that minimality is to be understood.

Our principal aim, in Theorems 4, 8, 9, 10, 11, 12 below, is to show that the context assumptions in [1, Theorem 2], [3], [6, Theorem 1] are not minimal and to characterize topological vector spaces that offer the proper context for an operator to be locally bounded, for example, on the algebraic interior of its domain convex hull.

The plan of the paper is as follows. In the next section we introduce the main notions and notations followed by a study of the so called Banach-Steinhaus property. The main object of Sections 3 and 4 is to provide characterizations of the topological vector spaces on which the local boundedness of an operator holds on different subsets directly related to the operator via its Fitzpatrick function.

## 2 Preliminaries

In this paper the conventions  $\sup \emptyset = -\infty$  and  $\inf \emptyset = \infty$  are enforced.

Given  $(E, \mu)$  a real topological vector space (TVS for short) and  $A \subset E$  we denote by “ $\text{conv } A$ ” the *convex hull* of  $A$ , “ $\text{cl}_\mu(A) = \overline{A}^\mu$ ” the  $\mu$ -*closure* of  $A$ , “ $\text{int}_\mu A$ ” the  $\mu$ -*topological interior* of  $A$ , “ $\text{core } A$ ” the *algebraic interior* of  $A$ . The use of the  $\mu$ -notation is avoided whenever the topology  $\mu$  is implicitly understood.

We denote by  $\iota_A$  the *indicator function* of  $A \subset E$  defined by  $\iota_A(x) := 0$  for  $x \in A$  and  $\iota_A(x) := \infty$  for  $x \in E \setminus A$ .

For  $f, g : E \rightarrow \overline{\mathbb{R}}$  we set  $[f \leq g] := \{x \in E \mid f(x) \leq g(x)\}$ ; the sets  $[f = g]$ ,  $[f < g]$ , and  $[f > g]$  being defined in a similar manner.

Throughout this paper, if not otherwise explicitly mentioned,  $(X, \tau)$  is a non-trivial (that is,  $X \neq \{0\}$ ) TVS,  $X^*$  is its topological dual endowed with the weak-star topology  $w^*$ , the topological dual of  $(X^*, w^*)$  is identified with  $X$  and the weak topology on  $X$  is denoted by  $w$ . The *duality product* of  $X \times X^*$  is denoted by  $\langle x, x^* \rangle := x^*(x) =: c(x, x^*)$ , for  $x \in X$ ,  $x^* \in X^*$ .

The class of neighborhoods of  $x \in X$  in  $(X, \tau)$  is denoted by  $\mathcal{V}_\tau(x)$ .

As usual, with respect to the dual system  $(X, X^*)$ , for  $A \subset X$ , the *orthogonal of*  $A$  is  $A^\perp := \{x^* \in X^* \mid \langle x, x^* \rangle = 0, \forall x \in A\}$ , the *polar of*  $A$  is  $A^\circ := \{x^* \in X^* \mid |\langle x, x^* \rangle| \leq 1, \forall x \in A\}$ , the *support function of*  $A$  is  $\sigma_A(x^*) := \sup_{x \in A} \langle x, x^* \rangle$ ,  $x^* \in X^*$  while for  $B \subset X^*$ , the *orthogonal of*  $B$  is  $B^\perp := \{x \in X \mid \langle x, x^* \rangle = 0, \forall x^* \in B\}$ , the *polar of*  $B$  is  $B^\circ := \{x \in X \mid |\langle x, x^* \rangle| \leq 1, \forall x^* \in B\}$ , and the *support function of*  $B$  is  $\sigma_B(x) := \sup_{x^* \in B} \langle x, x^* \rangle$ ,  $x \in X$ .

To a multifunction  $T : X \rightrightarrows X^*$  we associate its *graph*:  $\text{Graph } T = \{(x, x^*) \in X \times X^* \mid x^* \in Tx\}$ , *domain*:  $D(T) := \{x \in X \mid Tx \neq \emptyset\} = \text{Pr}_X(\text{Graph } T)$ , and *range*:  $R(T) := \{x^* \in X^* \mid x^* \in T(x), \text{ for some } x \in X\} = \text{Pr}_{X^*}(\text{Graph } T)$ . Here  $\text{Pr}_X$  and  $\text{Pr}_{X^*}$  are the projections of  $X \times X^*$  onto  $X$  and  $X^*$ , respectively. When no confusion can occur,  $T$  will be identified

with  $\text{Graph } T$ .

The *Fitzpatrick function* associated to  $T : X \rightrightarrows X^*$ ,  $\varphi_T : X \times X^* \rightarrow \overline{\mathbb{R}}$  is given by (see [4])

$$\varphi_T(x, x^*) := \sup\{\langle a, x^* \rangle + \langle x - a, a^* \rangle \mid a^* \in Ta\}, \quad (x, x^*) \in X \times X^*.$$

Accordingly, for every  $\epsilon \in \mathbb{R}$ , the set  $T_\epsilon^+ := [\varphi_T \leq c + \epsilon]$  describes all  $(x, x^*) \in X \times X^*$  that are  $\epsilon$ -monotonically related (m.r. for short) to  $T$ , that is  $(x, x^*) \in [\varphi_T \leq c + \epsilon]$  iff  $\langle x - a, x^* - a^* \rangle \geq -\epsilon$ , for every  $(a, a^*) \in T$ .

For every  $\epsilon \geq 0$ , we consider on a TVS  $(X, \tau)$  the following classes of functions and operators

$\Lambda(X)$  the class formed by proper convex functions  $f : X \rightarrow \overline{\mathbb{R}}$ . Recall that  $f$  is *proper* if  $\text{dom } f := \{x \in X \mid f(x) < \infty\}$  is nonempty and  $f$  does not take the value  $-\infty$ ,

$\Gamma_\tau(X)$  the class of functions  $f \in \Lambda(X)$  that are  $\tau$ -lower semi-continuous ( $\tau$ -lsc for short),

$\mathcal{M}_\epsilon(X)$  the class of non-empty  $\epsilon$ -monotone operators  $T : X \rightrightarrows X^*$ . Recall that  $T : X \rightrightarrows X^*$  is  $\epsilon$ -monotone if  $\langle x_1 - x_2, x_1^* - x_2^* \rangle \geq -\epsilon$ , for all  $(x_1, x_1^*), (x_2, x_2^*) \in T$ ,

$$\mathcal{M}_\epsilon^+(X) := \{T_\epsilon^+ \mid T \in \mathcal{M}_\epsilon(X)\},$$

$\mathfrak{M}_\epsilon(X)$  the class of  $\epsilon$ -maximal monotone operators  $T : X \rightrightarrows X^*$ . The maximality is understood in the sense of graph inclusion as subsets of  $X \times X^*$ ,

$$\mathcal{M}_\infty(X) := \bigcup_{\epsilon \geq 0} \mathcal{M}_\epsilon(X) = \{T : X \rightrightarrows X^* \mid \inf_{z, w \in T} c(z - w) \neq \pm\infty\},$$

the  $\epsilon$ -subdifferential of  $f$  at  $x \in X$ :  $\partial_\epsilon f(x) := \{x^* \in X^* \mid \langle x' - x, x^* \rangle + f(x) \leq f(x') + \epsilon, \forall x' \in X\}$  for  $x \in \text{dom } f$ ;  $\partial_\epsilon f(x) := \emptyset$  for  $x \notin \text{dom } f$ ,

$$\mathcal{G}_\epsilon(X) := \{\partial_\epsilon f \mid f \in \Gamma_\tau(X)\}, \quad \mathfrak{B}(X) := \{\partial\sigma_B \mid B \subset X^* \text{ is } w^*\text{-bounded}\}.$$

For  $\epsilon = 0$ , the use of the  $\epsilon$ -notation is avoided.

**Definition 1** Let  $(X, \tau)$  be a TVS. A multi-function  $T : X \rightrightarrows X^*$  is *locally bounded at*  $x_0 \in X$  if there exists  $U \in \mathcal{V}_\tau(x_0)$  such that  $T(U) := \bigcup_{x \in U} Tx$  is an equicontinuous subset of  $X^*$ ; *locally bounded on*  $S \subset X$  if  $T$  is locally

bounded at every  $x \in S$ . The local boundedness of  $T : X \rightrightarrows X^*$  is interesting only at  $x_0 \in \overline{D(T)}$  ( $\tau$ -closure) since for every  $x_0 \notin \overline{D(T)}$  there is  $U \in \mathcal{V}_\tau(x_0)$  such that  $T(U)$  is void. Consequently,  $T$  is locally bounded outside  $\overline{D(T)}$ .

Given a TVS  $(X, \tau)$  with topological dual  $X^*$ , a set  $B \subset X^*$  is:

- *pointwise-bounded* if  $B_x := \{x^*(x) \mid x^* \in B\}$  is bounded in  $\mathbb{R}$ , for every  $x \in X$  or, equivalently,  $B$  is  $w^*$ -bounded in  $X^*$ ;
- $(\tau)$ -*equicontinuous* if for every  $\epsilon > 0$  there is  $V_\epsilon \in \mathcal{V}_\tau(0)$  such that  $x^*(V_\epsilon) \subset (-\epsilon, \epsilon)$ , for every  $x^* \in B$ , or, equivalently,  $B$  is contained in the polar  $V^\circ$  of some (symmetric)  $V \in \mathcal{V}_\tau(0)$ .

We say that a topological vector space  $(X, \tau)$  has the *Banach-Steinhaus property* if every pointwise-bounded subset of  $X^*$  is equicontinuous.

**Theorem 2** *Let  $(X, \tau)$  be a TVS. The following are equivalent*

- (i)  $(X, \tau)$  has the Banach-Steinhaus property.
- (ii) Every absorbing, convex, and weakly-closed subset of  $X$  is a  $\tau$ -neighborhood of  $0 \in X$ .
- (iii) Every  $f \in \Gamma_w(X)$  is  $\tau$ -continuous on  $\text{int}_\tau(\text{dom } f)$  (or, equivalently,  $f$  is bounded above on a  $\tau$ -neighborhood of some  $x \in \text{dom } f$ ). In this case, for every  $f \in \Gamma_\tau(X)$ ,  $\text{int}_\tau(\text{dom } f) = \text{core}(\text{dom } f)$ .

**Proof.** (i)  $\Rightarrow$  (ii) Let  $C \subset X$  be absorbing, convex, and weakly-closed. Then  $C^\circ$  is pointwise-bounded and equicontinuous in  $X^*$  due to the Banach-Steinhaus property. Therefore  $C^\circ \subset V^\circ$ , for some  $V \in \mathcal{V}_\tau(0)$  followed by  $V \subset V^{\circ\circ} \subset C^{\circ\circ} = C$ , due to the Bipolar Theorem, and so  $C \in \mathcal{V}_\tau(0)$ .

(ii)  $\Rightarrow$  (iii) Let  $f \in \Gamma_w(X)$ ,  $x_0 \in \text{core}(\text{dom } f)$ , and  $a > f(x_0)$ . The level set  $[f \leq a]$  is weakly-closed and convex.

For every  $x \in X$ , let  $\mu > 0$  be such that  $\mu x \in \text{dom } f - x_0$ . Therefore, for every  $0 \leq \lambda \leq 1$ ,  $f(x_0 + \lambda\mu x) = f(\lambda(x_0 + \mu x) + (1 - \lambda)x_0) \leq \lambda f(x_0 + \mu x) + (1 - \lambda)f(x_0) = f(x_0) + \lambda(f(x_0 + \mu x) - f(x_0))$ .

Pick  $\lambda > 0$  sufficiently small to have  $f(x_0) + \lambda(f(x_0 + \mu x) - f(x_0)) \leq a$ . Hence  $x_0 + \lambda\mu x \in [f \leq a]$ , that is,  $[f \leq a] - x_0$  is absorbing. This implies  $[f \leq a] \in \mathcal{V}_\tau(x_0)$  and so  $x_0 \in \text{int}_\tau(\text{dom } f)$ . It is clear that  $f$  is bounded above on  $[f \leq a]$ .

(iii)  $\Rightarrow$  (i) If  $B \subset X^*$  is pointwise-bounded then  $\text{dom } \sigma_B = X$  and  $\sigma_B \in \Gamma_w(X)$ . Thus  $\sigma_B$  is  $\tau$ -continuous at 0, i.e., there exist a symmetric  $V \in \mathcal{V}_\tau(0)$  and  $M < \infty$  such that  $\sigma_B(x) \leq M$ , for every  $x \in V$ . This comes to  $B \subset (\frac{1}{M}V)^\circ$ , that is,  $B$  is  $\tau$ -equicontinuous. ■

**Remark 3** When  $(X, \tau)$  is a locally convex space (LCS for short), the closed convex sets in the  $\tau$  and weak topologies on  $X$  coincide. In this case the Banach-Steinhaus property comes to the fact that  $(X, \tau)$  is *barreled*, that is, every absorbing, convex, and  $\tau$ -closed subset of  $X$  is a  $\tau$ -neighborhood of  $0 \in X$ . Equivalently, every  $f \in \Gamma_\tau(X)$  is  $\tau$ -continuous on  $\text{int}_\tau(\text{dom } f)$ , in which case,  $\text{int}_\tau(\text{dom } f) = \text{core}(\text{dom } f)$ .

For every TVS  $(X, \tau)$ , let us denote by  $\tau^\circ$  the weakest local convex topology on  $X$  which is compatible with the duality  $(X, X^*)$  and finer than  $\tau$ . In case  $\tau^\circ$  exists,  $\tau$  and  $\tau^\circ$  share the equicontinuous sets of  $X^*$  and, in general, all the properties relying on equicontinuity or duality do not distinguish themselves between  $(X, \tau)$  and  $(X, \tau^\circ)$ . From this point of view it is the same if we consider the TVS  $(X, \tau)$  or its associated LCS  $(X, \tau^\circ)$ .

In the pathological cases when  $\tau^\circ$  does not exist, e.g. when  $X^* = \{0\}$  and  $(X, \tau)$  is (Hausdorff) separated, the operator local boundedness is trivially verified.

### 3 The local boundedness theorem

Let us note that in every TVS  $X$  there exist maximal monotone operators  $T : X \rightrightarrows X^*$  such that  $\text{int } \text{Pr}_X(\text{dom } \varphi_T)$  is non-empty. For example  $T = X \times \{0\}$  has  $\varphi_T(x, x^*) = \iota_{\{0\}}(x^*)$ ,  $(x, x^*) \in X \times X^*$  and  $\text{dom } \varphi_T = X \times \{0\}$ . This example is the most general possible since there exist non-trivial TVS's  $X$  for which  $X^* = \{0\}$ .

Our next result proves that the Banach-Steinhaus property is the minimal context condition under which the local boundedness of an operator with proper Fitzpatrick function holds.

**Theorem 4** *Let  $X$  be a TVS. The following are equivalent:*

- (i)  *$X$  has the Banach-Steinhaus property;*
- (ii) *Every  $T : X \rightrightarrows X^*$  is locally bounded on  $\text{core } \text{Pr}_X(\text{dom } \varphi_T)$ ;*
- (iii) *Every  $T : X \rightrightarrows X^*$  is locally bounded on  $\text{int } \text{Pr}_X(\text{dom } \varphi_T)$ .*

**Proof.** (i)  $\Rightarrow$  (ii) see the published version.

(ii)  $\Rightarrow$  (iii) is straightforward.

(iii)  $\Rightarrow$  (i) Let  $B \subset X^*$  be pointwise-bounded and let  $T : X \rightrightarrows X^*$  be such that  $\text{Graph } T = \{0\} \times B$ . Then  $\text{dom } \varphi_T = X \times X^*$  since  $\varphi_T(x, x^*) = \sigma_B(x) < \infty$ , for every  $x \in X$ ,  $x^* \in X^*$ . The local boundedness of  $T$  at

0 shows that  $T0 = B$  is equicontinuous, i.e.,  $X$  has the Banach-Steinhaus property. ■

As previously seen in Remark 3, when  $(X, \tau)$  is a LCS, condition (i) in Theorem 4 can be equivalently rephrased as  $(X, \tau)$  is barreled.

In the previous result, only the operators which have a proper Fitzpatrick function are interesting. We denote this class by

$$\mathcal{P}(X) := \{T : X \rightrightarrows X^* \mid \text{Graph } T \neq \emptyset, \text{ dom } \varphi_T \neq \emptyset\}.$$

In the literature, the most used class of operators that have a proper Fitzpatrick function is the class of non-empty monotone operators (see e.g. [7, 8, 9, 10, 12, 13, 14]), but, more generally, it is easily checked that  $\mathcal{M}_\infty(X) \subset \mathcal{P}(X)$  since  $\text{Graph } T \subset \text{dom } \varphi_T$ , for every  $T \in \mathcal{M}_\infty(X)$ .

**Definition 5** Given  $(X, \tau)$  a TVS, for every  $T : X \rightrightarrows X^*$  we denote by

$$\Omega_T := \{x \in \overline{D(T)} \mid T \text{ is locally bounded at } x\}$$

the (*meaningful*) *local boundedness set* of  $T$ .

In this notation, Theorem 4 states that the TVS  $(X, \tau)$  has the Banach-Steinhaus property iff

$$\text{core Pr}_X(\text{dom } \varphi_T) \cap \overline{D(T)} \subset \Omega_T, \quad \forall T \in \mathcal{P}(X) \quad (1)$$

iff

$$\text{int Pr}_X(\text{dom } \varphi_T) \cap \overline{D(T)} \subset \Omega_T, \quad \forall T \in \mathcal{P}(X). \quad (2)$$

On one hand, one cannot expect that, for every  $T \in \mathcal{P}(X)$ , the inclusions in (1) or (2) to be equalities, since there exist operators  $T$  which are locally bounded at some  $x \in D(T)$  but  $x \notin \text{core Pr}_X(\text{dom } \varphi_T)$  simply because  $\text{core Pr}_X(\text{dom } \varphi_T)$  is empty. Indeed, take  $X$  a Hilbert space,  $V \in \mathcal{V}(0)$ ,  $M \subset X$  a proper closed subspace, and  $T : D(T) = M \subset X \rightrightarrows X^*$ ,  $Tx = \{0\}$ , if  $x \in M \cap V$ ;  $Tx = M^\perp$ ,  $x \in M \setminus V$ . Then  $\varphi_T = \iota_{M \times M^\perp}$  and so  $\text{Pr}_X(\text{dom } \varphi_T) = M$  and  $\text{core Pr}_X(\text{dom } \varphi_T) = \emptyset$ .

Therefore, for some  $T \in \mathcal{P}(X)$ , the algebraic (or topological) interior of  $\text{Pr}_X(\text{dom } \varphi_T)$  is not the perfect description for  $\Omega_T$ .

On the other hand, for  $X$  a Banach space and  $T \in \mathfrak{M}(X)$  with  $\overline{D(T)}$  convex

$$\text{int Pr}_X(\text{dom } \varphi_T) = \text{core Pr}_X(\text{dom } \varphi_T) = \text{int } D(T) = \Omega_T, \quad (3)$$

(see [15, Theorem 3.11.15, p. 286], [6], and [11, Lemma 41]), that is, for this particular class of operators and type of space the problem of perfectly describing the local boundedness set is solved.

These two points of view prove that the general description of  $\Omega_T$ , given in Theorem 4 and provided for all operators  $T \in \mathcal{P}(X)$ , cannot be further improved.

We conjecture, that, under the assumption that  $X$  has the Banach-Steinhaus property,

$$\text{int Pr}_X(\text{dom } \varphi_T) \cap \overline{D(T)} = \text{core Pr}_X(\text{dom } \varphi_T) \cap \overline{D(T)}, \quad (4)$$

for every  $T \in \mathcal{P}(X)$  (or a suitable subclass such as  $\mathfrak{M}(X)$ ). A partial answer to this conjecture follows next.

**Proposition 6** *Let  $X$  be a barrelled normed space and let  $T : X \rightrightarrows X^*$ . Then  $\text{int Pr}_X(\text{dom } \varphi_T) = \text{core Pr}_X(\text{dom } \varphi_T)$ .*

**Proof.** Assuming that  $\text{core Pr}_X(\text{dom } \varphi_T) \neq \emptyset$  and because  $X$  is a barrelled normed space, we get the conclusion (see e.g. [15, Proposition 2.7.2 (vi), p. 116]). ■

## 4 Local boundedness on subclasses

This section deals with the validity of the implications (ii)  $\Rightarrow$  (i) or (iii)  $\Rightarrow$  (i) in Theorem 4 on subclasses of operators and subsets of local boundedness.

**Proposition 7** *Let  $(X, \tau)$  be a LCS and let  $B \subset X^*$  be pointwise-bounded. Then  $\{0\} \times B$  admits a (maximal) monotone extension  $T : X \rightrightarrows X^*$  with  $0 \in \text{core}(\text{conv } D(T))$ .*

**Proof.** see the published version. ■

**Theorem 8** *Let  $(X, \tau)$  be a TVS. For every  $\mathcal{C} \in \{\mathfrak{B}(X), \mathcal{G}(X), \mathcal{M}(X)\} \cup \{\mathcal{G}_\epsilon(X), \mathcal{M}_\epsilon(X), \mathfrak{M}_\epsilon(X), \mathcal{M}_\epsilon^+(X) \mid \epsilon > 0\}$  the following are equivalent*

- (i)  $(X, \tau)$  has the Banach-Steinhaus property,
- (ii)  $\text{core Pr}_X(\text{dom } \varphi_T) \cap \overline{D(T)} \subset \Omega_T, \forall T \in \mathcal{C},$
- (iii)  $\text{int Pr}_X(\text{dom } \varphi_T) \cap \overline{D(T)} \subset \Omega_T, \forall T \in \mathcal{C}.$

**Proof.** (ii)  $\Rightarrow$  (iii) is plain. (i)  $\Rightarrow$  (ii) holds due to Theorem 4 because every considered class of operators  $\mathcal{C}$  is a subclass of  $\mathcal{P}(X)$ . More precisely, if  $M \in \mathcal{M}_\epsilon^+(X)$ , for some  $\epsilon \geq 0$ , then,  $M = T_\epsilon^+$ , for some  $T \in \mathcal{M}_\epsilon(X)$ . We have  $T \subset [\varphi_M \leq c + \epsilon] \subset \text{dom } \varphi_M$ ; whence  $M \in \mathcal{P}(X)$  and so  $\bigcup_{\epsilon \geq 0} \mathcal{M}_\epsilon^+(X) \subset \mathcal{P}(X)$ .

(iii)  $\Rightarrow$  (i) For every pointwise bounded  $B \subset X^*$  we show that  $\{0\} \times B$  admits an extension  $T$  which belongs to every class considered and  $\text{Pr}_X(\text{dom } \varphi_T) = X$ . In this case the local boundedness of  $T$  at 0 proves that  $B$  is equicontinuous.

First, note that, whenever  $B \subset X^*$  is pointwise bounded,  $\text{dom } \sigma_B = X$  and from  $\varphi_{\partial \sigma_B}(x, x^*) \leq \sigma_B(x) + \iota_B(x^*)$ , for every  $(x, x^*) \in X \times X^*$ , one gets  $X \times B \subset \text{dom } \varphi_{\partial \sigma_B}$  and  $\text{Pr}_X(\text{dom } \varphi_{\partial \sigma_B}) = X$ . Therefore  $\partial \sigma_B \in \mathfrak{B}(X)$  fulfills all the required conditions. Since  $\mathfrak{B}(X) \subset \mathcal{G}(X) \subset \mathcal{M}(X)$ , this example completes the argument for the classes  $\mathfrak{B}(X), \mathcal{G}(X), \mathcal{M}(X)$ .

Also, since  $\sigma_B \in \Gamma_\tau(X)$  and  $\text{dom } \sigma_B = X$ ,  $D(\partial_\epsilon \sigma_B) = X$ , for every  $\epsilon > 0$ ; whence  $\partial_\epsilon \sigma_B \in \mathcal{G}_\epsilon(X)$  has the required properties. Since  $\mathcal{G}_\epsilon(X) \subset \mathcal{M}_\epsilon(X)$ ,  $D(T) = X$ , for every extension  $T$  of  $\partial_\epsilon \sigma_B$ , and  $\mathfrak{M}_\epsilon(X) \subset \mathcal{M}_\epsilon^+(X)$  this example proves the implication for the classes  $\{\mathcal{G}_\epsilon(X), \mathcal{M}_\epsilon(X), \mathfrak{M}_\epsilon(X), \mathcal{M}_\epsilon^+(X) \mid \epsilon > 0\}$ . ■

**Theorem 9** *Let  $(X, \tau)$  be a LCS. For every  $\mathcal{C} \in \{\mathfrak{M}(X), \mathcal{M}^+(X)\}$  the following are equivalent*

- (i)  $(X, \tau)$  is barreled,
- (ii)  $\text{core Pr}_X(\text{dom } \varphi_T) \cap \overline{D(T)} \subset \Omega_T, \forall T \in \mathcal{C}.$

**Proof.** Since  $\mathfrak{M}(X) \cup \mathcal{M}^+(X) \subset \mathcal{P}(X)$ , (i)  $\Rightarrow$  (ii) holds due to Theorem 4.

We have seen in Proposition 7, that  $\{0\} \times B$  admits a maximal monotone extension  $T : X \rightrightarrows X^*$  with  $0 \in \text{core}(\text{conv } D(T)) \subset \text{core Pr}_X(\text{dom } \varphi_T)$ . Together with  $\mathfrak{M}(X) \subset \mathcal{M}^+(X)$  we infer that (ii)  $\Rightarrow$  (i) is true for  $\mathcal{C} \in \{\mathfrak{M}(X), \mathcal{M}^+(X)\}$ . ■



**Theorem 10** *Let  $(X, \tau)$  be a TVS. For every  $\mathcal{C} \in \{\mathcal{G}_\epsilon(X), \mathcal{M}_\epsilon(X), \mathfrak{M}_\epsilon(X), \mathcal{M}_\epsilon^+(X) \mid \epsilon > 0\}$  the following are equivalent*

- (i)  $(X, \tau)$  has the Banach-Steinhaus property,
- (ii)  $\text{core}(\text{conv } D(T)) \cap \overline{D(T)} \subset \Omega_T, \forall T \in \mathcal{C}.$
- (iii)  $\text{int}(\text{conv } D(T)) \cap \overline{D(T)} \subset \Omega_T, \forall T \in \mathcal{C}.$

**Proof.** (ii)  $\Rightarrow$  (iii) is plain. (i)  $\Rightarrow$  (ii) For  $\mathcal{C} \in \{\mathcal{G}_\epsilon(X), \mathcal{M}_\epsilon(X), \mathfrak{M}_\epsilon(X) \mid \epsilon > 0\}$  we use Theorem 4, because all these classes are subclasses of  $\mathcal{M}_\infty(X)$  and  $D(T) \subset \text{Pr}_X(\text{dom } \varphi_T)$ , for every  $T \in \mathcal{M}_\infty(X)$ .

For  $\mathcal{C} = \mathcal{M}_\epsilon^+(X)$ , let  $M \in \mathcal{M}_\epsilon^+(X)$ , i.e.,  $M = T_\epsilon^+$ , for some  $T \in \mathcal{M}_\epsilon(X)$ . Since  $T \in \mathcal{M}_\epsilon(X)$ ,  $T \subset M$  and  $T \subset \text{Pr}_X(\text{dom } \varphi_M)$ . We apply Theorem 4 for  $M$  to get that  $M$  and implicitly  $T$  are local bounded on  $\text{core } \text{Pr}_X(\text{dom } \varphi_M) \supset \text{core}(\text{conv } D(T))$ .

(iii)  $\Rightarrow$  (i) For every pointwise bounded  $B \subset X^*$  we claim that  $\{0\} \times B$  admits an extension  $T$  which belongs to every class considered and  $D(T) = X$ . In this case the local boundedness of  $T$  at 0 proves that  $B$  is equicontinuous.

Indeed, for  $\epsilon > 0$ ,  $\partial_\epsilon \sigma_B$  is an extension of  $\{0\} \times B$  which belongs to  $\mathcal{G}_\epsilon(X)$  and  $D(\partial_\epsilon \sigma_B) = X$ . Since  $\mathcal{G}_\epsilon(X) \subset \mathcal{M}_\epsilon(X)$ ,  $\mathfrak{M}_\epsilon(X) \subset \mathcal{M}_\epsilon^+(X)$ , and any extension  $T$  of  $\partial_\epsilon \sigma_B$  has  $D(T) = X$ , this example completes the argument for  $\{\mathcal{G}_\epsilon(X), \mathcal{M}_\epsilon(X), \mathfrak{M}_\epsilon(X), \mathcal{M}_\epsilon^+(X) \mid \epsilon > 0\}$ . ■

**Theorem 11** *Let  $(X, \tau)$  be a LCS. For every  $\mathcal{C} \in \{\mathfrak{M}(X), \mathcal{M}(X), \mathcal{M}^+(X)\}$  the following are equivalent*

- (i)  $(X, \tau)$  is barreled,
- (ii)  $\text{core}(\text{conv } D(T)) \cap \overline{D(T)} \subset \Omega_T, \forall T \in \mathcal{C}.$

**Proof.** From Theorem 4, (i)  $\Rightarrow$  (ii) holds since  $\mathcal{M}(X) \cup \mathcal{M}^+(X) \subset \mathcal{P}(X)$ .

Because  $\{0\} \times B$  admits a maximal monotone extension  $T : X \rightrightarrows X^*$  with  $0 \in \text{core}(\text{conv } D(T))$  (see Proposition 7), (ii)  $\Rightarrow$  (i) is true for  $\mathfrak{M}(X)$ , followed by its super-classes  $\mathcal{M}(X)$  and  $\mathcal{M}^+(X)$ . ■

The following theorem is a broad generalization of the main result in [1] and of [2, Theorem 4.2].

**Theorem 12** *Let  $(X, \tau)$  be a LCS and let  $\epsilon \geq 0$ . The following are equivalent:*

- (i)  $X$  is barreled,
- (ii) For every  $T \in \mathcal{M}_\epsilon(X)$ ,  $T_\epsilon^+$  is locally bounded on  $\text{core}(\text{conv } D(T))$ ;
- (iii) For every  $T \in \mathfrak{M}_\epsilon(X)$ ,  $T$  is locally bounded on  $\text{core}(\text{conv } D(T))$ .

**Proof.** (i)  $\Rightarrow$  (ii) follows from Theorem 4 because  $D(T) \subset \text{Pr}_X(\text{dom } \varphi_{T_\epsilon^+})$ , for every  $\epsilon \geq 0$ ,  $T \in \mathcal{M}_\epsilon(X)$ .

(ii)  $\Rightarrow$  (iii) Whenever  $T \in \mathfrak{M}_\epsilon(X)$ ,  $T = T_\epsilon^+$ .

(iii)  $\Rightarrow$  (i) We use Theorems 10, 11 to conclude. ■

## References

- [1] Jon Borwein and Simon Fitzpatrick. Local boundedness of monotone operators under minimal hypotheses. *Bull. Austral. Math. Soc.*, 39(3):439–441, 1989.
- [2] Jean-Pierre Crouzeix and Eladio Ocaña Anaya. Maximality is nothing but continuity. *J. Convex Anal.*, 17(2):521–534, 2010.
- [3] P. M. Fitzpatrick, P. Hess, and Tosio Kato. Local boundedness of monotone-type operators. *Proc. Japan Acad.*, 48:275–277, 1972.
- [4] Simon Fitzpatrick. Representing monotone operators by convex functions. In *Workshop/Miniconference on Functional Analysis and Optimization (Canberra, 1988)*, volume 20 of *Proc. Centre Math. Anal. Austral. Nat. Univ.*, pages 59–65. Austral. Nat. Univ., Canberra, 1988.
- [5] Tosio Kato. Demicontinuity, hemicontinuity and monotonicity. II. *Bull. Amer. Math. Soc.*, 73:886–889, 1967.
- [6] R. T. Rockafellar. Local boundedness of nonlinear, monotone operators. *Michigan Math. J.*, 16:397–407, 1969.
- [7] M. D. Voisei. A maximality theorem for the sum of maximal monotone operators in non-reflexive Banach spaces. *Math. Sci. Res. J.*, 10(2):36–41, 2006.
- [8] M. D. Voisei. Calculus rules for maximal monotone operators in general Banach spaces. *J. Convex Anal.*, 15(1):73–85, 2008.

- [9] M. D. Voisei. The sum and chain rules for maximal monotone operators. *Set-Valued Anal.*, 16(4):461–476, 2008.
- [10] M. D. Voisei. A sum theorem for (FPV) operators and normal cones. *J. Math. Anal. Appl.*, 371:661–664, 2010.
- [11] M. D. Voisei. Characterizations and continuity properties for maximal monotone operators with non-empty domain interior. *J. Math. Anal. Appl.*, 391:119–138, 2012.
- [12] M. D. Voisei and C. Zălinescu. Strongly-representable monotone operators. *J. Convex Anal.*, 16(3-4):1011–1033, 2009.
- [13] M. D. Voisei and C. Zălinescu. Linear monotone subspaces of locally convex spaces. *Set-Valued Var. Anal.*, 18(1):29–55, 2010.
- [14] M. D. Voisei and C. Zălinescu. Maximal monotonicity criteria for the composition and the sum under weak interiority conditions. *Math. Program.*, 123(1, Ser. B):265–283, 2010.
- [15] C. Zălinescu. *Convex analysis in general vector spaces*. World Scientific Publishing Co. Inc., River Edge, NJ, 2002.